

Multiplications and Proportions
Systems of Equations
Probability

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Grade Level: 7-12

Executive Summary

Unit 1 is on systems of equations and shows concrete examples of how algebra formulas relate to real life problems. The standards and MCA questions are built within the lessons as they go. When the students explore the systems of equations problems, they are given an initial problem to explore with the class. Once the class discovers how to solve the problem, the students then work in small groups so they can further discover different ways to solve the systems of equations. In unit 2, we get into different types of theoretical probability. Students will explore permutations and combinations by exploring real life examples and working in groups to solve the problems. In Unit 3 you will see an exploration dealing with looking at patterns to come up with an explicit formula for arithmetic and geometric sequences. There is also an exploration on finding how many squares are in an 8 x 8 square checkerboard. Unit 4 exploration deals with looking for patterns to find a recursive and explicit formula to find the area of irregular shapes. Unit 5 deals with the four voting methods we were taught in class and how students can look at the different methods and problem solve a solution.

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Unit 3

Recursive and Explicit Formula (arithmetic and geometric sequences)

Unit 4

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Lesson 2-Picks Formula

Unit 5

Winner, Winner

Unit 1: Systems of Equations

Lesson 1 Objective: Writing two linear equations

Materials needed: paper/pencil

Homework: 3 practice problems

Standard: represent, analyze, and generalize a variety of patterns with tables, graphs, words, and, when possible, symbolic rules; relate and compare different forms of representation for a relationship; identify functions as linear or nonlinear and contrast their properties from tables, graphs, or equations.

Number of Days: 4

Pre Test and Post Test:

Lets say there are a bin of spiders and ants. When looking I see 37 heads and 254 legs total. How many spiders and ants are there? Solve using substitution, by graphing, and elimination.

Day 1

Launch: I will tell a story about a dog park that is coming to town and how I have a dog that likes to run. I'll tell them about my dog Metani and her great qualities. Then I'll say we are at a dog park and need to find the amount of dogs and kids. There are 28 total legs and ten heads. How many dogs and people are there? We will do this one together as a class.

Answer: $4d+2k=28$ for total number of legs. $d+k=10$ for total number of heads. $d=4$, $k=6$

Explore: I will give them all a different problem with the same story. Legs=60 and heads=18

Students will work in groups of four on this problem. Answer: $4d+2k=60$, $d+k=18$

Share: Students will share how they arrived to their answer.

Summarize: I will make sure to point out how to set up the equations for these problems.

Homework: Students will work in groups again and work on setting up the equations for the next few problems.

Now there is an ant and spider farm and the students need to find the number of spiders and ants.

Problem 1: 5 heads and 36 legs

Answer: $8s+6a=36$, $s+a=5$

Problem 2: legs 206 and heads 30

Answer: $8s+6a=206$, $s+a=30$

Problem 3: legs 260 and heads 40

Answer: $8s+6a=260$, $s+a=40$

Day 2

Objective: Substitution

Launch: Let's look at the four problems we did yesterday and their answers. How long would it take to use guess and check method to find the number of dogs and kids. Are there other ways we can find the solutions to the problems? Then explore other algebraic methods.

Go over how to use substitution with the students to see how to use it with yesterday's equations $4d+2k=60$, $d+k=18$.

Explore: Have students use the substitution method for Problem 1: 5 heads and 36 legs

Answer: $8s+6a=36$, $s+a=5$

Share: Have students share how they arrived with the answers algebraically.

Summarize: Go over how to use substitution to solve systems of equations and make sure the students know the vocabulary.

Homework: Solve each problem using substitution

Problem 1: 5 heads and 36 legs

Answer: $8s+6a=36$, $s+a=5$

Problem 2: legs 206 and heads 30

Answer: $8s+6a=206$, $s+a=30$

Problem 3: legs 260 and heads 40

Answer: $8s+6a=260$, $s+a=40$

Day 3

Objective: Graphing

Launch: We will be looking at our same equations but looking at the different ways to display out formulas. Looking back to when you first got the problem using the guess and check method. Did you have different options for answers? Lets look at the first one in a table format. Answer: $4d+2k=28$ for total number of legs. $d+k=10$ for total number of heads. $d=4, k=6$

Total number of legs equations

d	k
2	10
3	8
4	6
5	4
6	2
7	0

Total number of heads

d	k
2	8
3	7
4	6
5	5
6	4
7	3

Then plot the points for both equations. What do you notice about the graph and table? They should find that the tables have the same point and the graphs intersect at the solution.

Explore: Students will graph this problem in pairs. Legs=60 and heads=18

Answer: $4d+2k=60$, $d+k=18$

Share: Students will share how they got their answers.

Summarize: I'll make sure they know how to set up the equations in slope intercept form and to graph them by making a table of values as well.

Homework: Solve by graphing.

Problem 1: 5 heads and 36 legs

Answer: $8s+6a=36$, $s+a=5$

Problem 2: legs 206 and heads 30

Answer: $8s+6a=206$, $s+a=30$

Problem 3: legs 260 and heads 40

Answer: $8s+6a=260$, $s+a=40$

Day 4

Objectives: Elimination

Launch: Looking back at when we solved using substitution. Is there another way to solve algebraically? Then we will talk about elimination with 28 total legs and ten heads.

Answer: $4d+2k=28$ for total number of legs. $d+k=10$ for total number of heads. $d=4$, $k=6$

Explore: Students will try elimination for themselves. 5 heads and 36 legs

Answer: $8s+6a=36$, $s+a=5$

Share: Students will share how they got their answers.

Summarize: I will make sure they understand the method of solving by elimination.

Homework: Solve by elimination

Problem 1: 5 heads and 36 legs

Answer: $8s+6a=36$, $s+a=5$

Problem 2: legs 206 and heads 30

Answer: $8s+6a=206$, $s+a=30$

Problem 3: legs 260 and heads 40

Answer: $8s+6a=260$, $s+a=40$

Unit 2 Pre Test and Post Test:

A standard deck of 52 playing cards is shuffled and a card is chosen at random. What is the probability that the card chosen is:

- a. A heart?
- b. A king?
- c. The king of hearts?
- d. A king or a heart?

[a: , b: , c: , d:]

A box contains toy train engines and box cars. There are two black engines, three blue engines and two red engines. Four green box cars and one yellow box car are also included in the box. If Seth reaches in without looking and chooses one engine and then one box car, what is the probability that he gets

- a. A red engine and a green box car?
- b. A black engine and a yellow box car?
- c. A blue engine and a green box car?
- d. Suppose Seth gets a blue engine and a green box car and now Daniel is going to choose one of each. What is the probability that Daniel also gets a blue engine and a green box car?

[a: , b: , c: , d:]

- o A card is drawn at random out of a 52 card deck.
- a. Find P (drawing a face card).
- b. Find P (drawing a heart).
- c. Find P (drawing a spade) if you know a black card was drawn.

[a: , b: , c:]

- o A penny, a nickel, a dime, and a quarter are all flipped.
- a. List all elements of the sample space (i.e. HHHH, . . .).
- b. What is the probability of getting **at least two tails**?
- c. What is the probability of getting **at least one head**?
- d. What is the probability of getting at least two tails if you already know the penny came up heads?

[a: HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, THTH, THHT, HTTH, TTHH, TTTH, TTHT, THTT, HTTT, TTTT; b: ; c: ; d:]

o A standard deck of 52 playing cards is shuffled and a card is chosen at random.

What is the probability that the card chosen is:

- a. A three?
- b. A black card?
- c. A three that is a black card?
- d. A three or any black card?

[a: , b: , c: , d:]

o Rebecca has an 98% probability of getting a problem correct on her homework, but only does about 84% of the problems assigned. How many problems would she get correct on an assignment with 25 problems? Show either a tree model or an area model to represent the situation.

[20 problems]

o A penny, a nickel, and a dime are all flipped.

- a. List all elements of the sample space (i.e. HHH, . . .).
- b. What is the probability of getting no tails?
- c. What is the probability of getting at least one tail?

[a: HHH, HHT, HTH, THH, TTT, TTH, THT, HTT; b: ; c:]

A jar contains 6 blue marbles, 7 red marbles, two green marbles, and ten yellow marbles.

- a. If you reach into the jar without looking, what is the probability of pulling out a red marble? a green marble?
- b. Davis pulled out a blue marble and kept it. Tess reached in. What is the probability she pulls out a blue marble as well? What is the probability that it is yellow instead?
- c. What is the probability of Aria reaching in and grabbing both green marbles?

[a: red, green; b: blue, yellow; c: or]

Unit 2: Probability and Counting

Lesson 1 Objective: Probability and Expected Value

Materials needed: paper/pencil

Homework: 10-10 to 10-17 and 10-18 to 10-26

Standard: compute basic statistics and understand the distinction between a statistic and a parameter.

Number of Days: 3

Day 1:

Launch: I'll have the students in the class do a paper, rock, scissors tournament. At the end I will ask them if the game was fair and we will discuss the results.

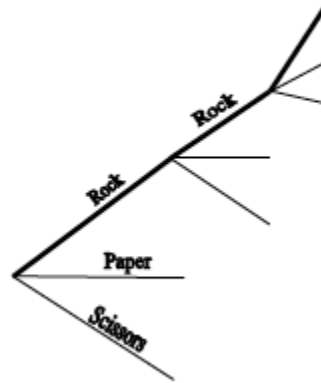
10-1. ROCK, PAPER, SCISSORS

Your team will play a variation of "Rock, Paper, Scissors" (sometimes called "Rochambeau") and record points. You will need to work in a team of four. Have one person act as recorder while the other three play the game. [c: no; ideas vary, d: There are three branches that represent all three players matching (RRR, PPP, SSS), $P(A) = 3 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$, e: $P(B) = 18 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{3}$, $P(C) = 6 \cdot \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$; Clearly, the game is not fair.]

- a. List the names of the people in your team alphabetically. The first person on the list is Player A, the next is Player B, the third is Player C, and the fourth is the recorder. Write down who has each role.
- b. Your team will play "Rock, Paper, Scissors." The recorder should record the winner for each round. Assign points as follows:
 - Player A gets a point each time all three players match.
 - Player B gets a point each time two of the three players match.
 - Player C gets a point each time none of the players match.

First, discuss with your team which player you think will receive the most points by the end of the game. Now, play "Rock, Paper, Scissors" at least 20 times.

- c. Does this game seem fair? How could you calculate the theoretical probabilities of each outcome (Player A, Player B, or Player C winning)? Discuss this with your team and be prepared to share your ideas with the class.
- d. Jenna is Player A in her team and she has decided to make a tree diagram to help her calculate the probability that she will win. The diagram she started is shown at right. Work with your team to complete this diagram. How many of the branches represent Jenna winning? Which ones? Calculate the probability that Jenna will win.
- e. Calculate the probabilities that Player B will win and that Player C will win. Is this a fair game?



Explore: Student will work in groups.

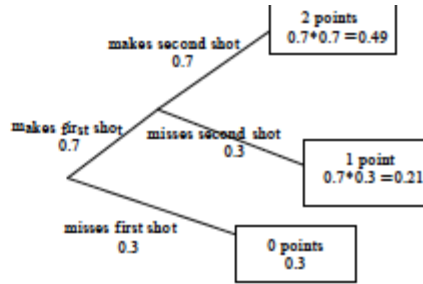
10-2. BASKETBALL: Shooting One-and-One Free Throws

Rimshot McGee has a 70% free throw average. The opposing team is ahead by one point. Rimshot is at the foul line in a one-and-one situation with just seconds left in the game. (A one-and-one situation means that the player shoots a free throw. If he makes the shot, he is allowed to shoot another. If he misses the shot, he gets no second shot. Each shot made is worth one point.) [a: 2 is most likely, but most will probably say 1. b: See tree diagram below. c: 1 point is represented by the upper left portion of the model. $P(1) = (0.7)(0.3) = 0.21$, d: $P(0) = 0.3$, $P(2) = (0.7)(0.7) = 0.49$, 2 points is most likely. e: $\frac{49(2)+21(1)+30(0)}{100} = 0.49(2) + 0.21(1) + 0.3(0) = 1.19$]

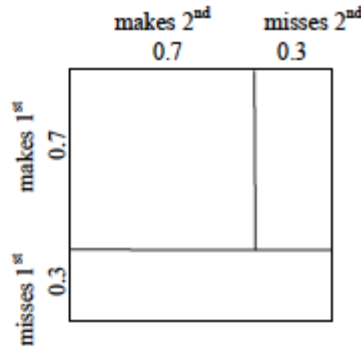


a. First, take a guess. What do you think is the most likely outcome for Rimshot: zero points, one point, or two points?

b. Draw a tree diagram to represent this situation.



c. Jeremy is working on the problem with Jenna and he remembers that area models are sometimes useful for solving problems related to probability. They set up the area model at right. Discuss this model with your team. Which part of the model represents Rimshot getting one point? How can you use the model to help calculate the probability that Rimshot will get exactly one point?



d. Use either your tree diagram or the area model to help you calculate the probabilities that Rimshot will get either 0 or 2 points. What is the most likely of the three outcomes?

e. Many people, when faced with this same problem, guess that one point is the most likely outcome. Their confusion comes from their intuitive understanding of expected value, which is the average number of points you would expect over many repetitions of the situation. In this case, if Rimshot McGee were in 100 one-and-one free throw situations, what would we expect to be his average number of points? Discuss this with your team and be ready to share your ideas with the class.

Share: The groups will share their answers for 10-2 and how they arrived at their answer. Then go on to work on 10-3 in groups.

10-3. With your team, take a minute to examine the area model for problem 10-2. [a: 1 unit by 1 unit; Each dimension represents probabilities of all of the possible outcomes of one of the two events. The sum of all of the outcomes should always be 1. b: 1 square unit; $(0.3 + 0.7)(0.3 + 0.7) = (0.7)(0.7) + (0.7)(0.3) + 1(0.3)$, c: The probability that Rimshot will score either 0, 1, or 2 points.]

- What are the dimensions of the rectangle? Explain why these dimensions make sense.
- What is the total area of the model? Express the area as a product of the dimensions and as a sum of the parts.
- What probability is represented by the entire area of the model?

Share: Students will share their answers again. 10-4 will be done with a partner and will be a formative assessment for the day.

10-4. DOUBLE SPIN

“Double Spin” is a new game at the fair. The player gets to spin a spinner twice, but wins only if the same amount comes up both times. The \$100 sector is $\frac{1}{8}$ of the circle. Make an area model or tree diagram to show the probabilities of every outcome of two spins and then answer the following questions. [See diagram below right.



a: $\frac{1}{64} \approx 1.56\%$, b: $\frac{1}{64} + \frac{2}{64} + \frac{1}{16} + \frac{1}{16} = \frac{18}{64} = 28.125\%$,
 c: $\frac{1}{16} \cdot \$10 + \frac{1}{16} \cdot \$5 + \frac{2}{64} \cdot \$1 + \frac{1}{64} \cdot \$100 + \frac{46}{64} \cdot \$0 \approx \$2.64$, d: no]

	1/8	3/8	1/4	1/4
1/8	1/64			
3/8		9/64		
1/4			1/16	
1/4				1/16

- What is the probability of winning \$100 ?
- What is the probability of winning anything at all?
- How did your team choose whether to use an area model or a tree diagram? Now that you have calculated the probabilities, do you think you made the right choice? Why or why not?
- What is the **expected value** for a winner of this game? That is, what is the average amount of money the carnival should expect to pay to players each turn over a long period of time?
- If it costs \$3.00 for you to play this game, should you expect to break even in the long run?

Day 2: Begin with Exploring new problems in groups.

- 10-5. Marty and Gerri had just lost playing “Double Spin” when they ran into another game they had never heard of called “Pick a Tile,” in which the player has to reach into a bag and choose one square tile and one circular tile. The bag contains three yellow, one blue and two red squares as well as one yellow and two red circles. In order to win the game (and a large stuffed animal), a player must choose one blue square and one red circle.

Since it costs \$2 to play the game, Marty and Gerri decided to calculate the probability of winning before deciding whether to play.

Gerri suggested making a list of all the possible color combinations, squares first then circles:

$RY \quad BY \quad YY$
 $RR \quad BR \quad YR$



“So,” says Gerri, “the answer is $\frac{1}{6}$.”

“That doesn’t seem quite right,” says Marty. “There are more yellow squares in there than blue ones. I don’t think the chance of getting a yellow square and a red circle should be the same as getting a blue square and a red circle. Maybe we need to account for all three yellow squares with Y_1, Y_2, Y_3 .”

- Make a tree diagram or a systematic list of all the possibilities, using subscripts to account for the colors for which there is more than one square or circle.
 [a: $Y_1Y, Y_2Y, Y_3Y, Y_1R_1, Y_1R_2, BY, BR_1, BR_2, R_1Y, R_1R_1, R_1R_2, R_2Y, R_2R_1, R_2R_2$; b: $P(BR) = \frac{1}{9}$; c: Students will probably say no because they have a $\frac{8}{9}$ chance of losing their money.]
- Find the probability of a player choosing the blue-red combination.
- Should Gerri and Marty play this game? Would you? Why or why not?

- 10-6. How could the area model at right help Marty and Gerri calculate all of the probabilities? [a: $P(YR) = \frac{1}{3}, P(YY) = \frac{1}{6}, P(BR) = \frac{1}{9}, P(BY) = \frac{1}{18}, P(RR) = \frac{2}{9}, P(RY) = \frac{1}{9}$, b: Ideas vary. Students may comment that the different number possibilities for each color are represented by different lengths along the respective side of the model.]

$P(R) = \quad P(Y) =$

- Complete the area model and use it to calculate the probability of each possible color combination of a square and a circular tile.
- Talk with your team about how this model is related to the systematic list or tree diagram that you made in problem 10-5. Find as many connections as you can and be prepared to share your ideas with the class.

$P(Y) =$

$P(R) =$

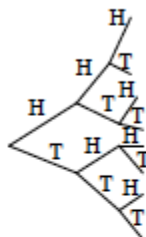
$P(B) =$

10-7. BASKETBALL: Shooting One-and-One Free Throws Revisited

Dunkin' Delilah Jones has a 60% free throw average. [a: 0 points; b: It is more likely that she will make some points, as $P(1 \text{ or } 2) = 0.36 + 0.24 = 0.6$ and $P(0) = 0.4$; c: $2(0.36) + 1(0.24) + 0(0.16) = 0.96$; d: possibilities include 50% average leads to most likely result of 0 points, 80% average leads to most likely result of 2 points, 65% average leads to most likely result of 2 points. e: 61.803%, f: $1 - x = x^2$; $x = \frac{-1 + \sqrt{5}}{2}$]

- What would be the most likely result when she shoots a one-and-one?
- Is it more likely that Delilah would make no points or that she would score some points? Explain.
- On average, how many points would you expect Dunkin' Delilah to make in a one and one free throw situation? That is, what is the expected value?
- Try at least three other possible free throw percentages and make a note of the most likely outcome.
- Is there some free-throw percentage that would make two points and zero points equally likely outcomes? If so, find this percentage.
- If you did not already do so, draw an area model or tree diagram for part (e) using x as the percentage and write an equation to represent the problem. Write the solution to the equation in simplest radical form.

10-8. Suppose that you were going to flip three coins: a penny, a nickel, and a dime. [a: HHH, HHT, HTH, HTT, THH, THT, TTH, TTT, no; b: 8; c: $i: \frac{1}{8}$, $ii: \frac{1}{2}$, $iii: \frac{3}{8}$, $iv: \frac{7}{8}$, $v: \frac{3}{8}$, $vi: \frac{3}{4}$; d: They are the same. e: See tree diagram below right. An area model does not make sense because it would need a third dimension.]



- Make a systematic list that shows all the possible outcomes. This list of all the possible outcomes is known as the sample space for this experiment. Does it matter which coin is flipped first?
- How many outcomes are there?
- Find the probability of flipping each of the following. Be sure to show your thinking clearly:

<i>i.</i> Three heads	<i>ii.</i> At least two heads
<i>iii.</i> One head and two tails	<i>iv.</i> At least one tail
<i>v.</i> Exactly two tails	<i>vi.</i> At least one head and one tail
- Which is more likely, flipping at least 2 heads or at least 2 tails? Explain.
- If you have not done so already, represent this sample space with an area model or a tree diagram. Is there one model that makes more sense than the other in this situation? Explain.

- 10-9. For what kinds of probability problems might area models be most useful? What about tree diagrams? Are there any types of problems in which either model would be impossible to use? Explain. [Answers vary. Students should realize that area models are only useful with two sets of outcomes to consider. They may comment that tree diagrams get too big when there are too many options at each branch.]

Share: Each group will go over one of the problems and how they arrived to their answers.
 Summarize: I will make sure to point out the major points that were not mentioned.



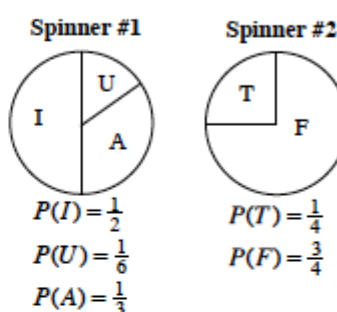
MATH NOTES

METHODS AND MEANINGS

When all the possible outcomes of an event are **equally likely** (in other words have the same probability), the probability of a sequence of such events can be found by making a simple **systematic list**. However, when some outcomes are more likely than others, a more sophisticated model is required.

One such model is an **area model**. In this type of model, the situation is represented by a square with dimensions 1 X 1 so that the areas of the parts are the probabilities of the different events that can occur. For example, suppose you spin the two spinners shown above. The probabilities of the outcomes are represented as the lengths along the sides of the area model at right. Notice that the "U" column takes up $\frac{1}{6}$ of the width of the table since $P(U) = \frac{1}{6}$. Similarly, the "T" row takes up $\frac{1}{4}$ of the height of the table, since $P(T) = \frac{1}{4}$. Then the probability that the spinners come out "U" and "T" is equal to the area of the $P(U) \times P(T)$ rectangle in the table:
 $\frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$.

Probability Models

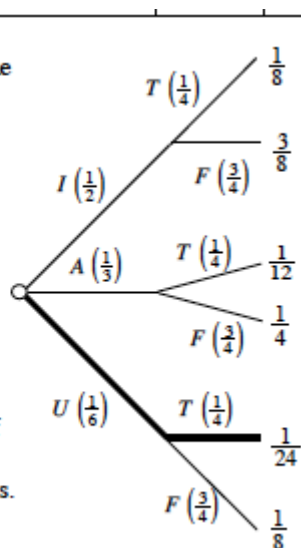


		Spinner #1		
		I $(\frac{1}{2})$	A $(\frac{1}{3})$	U $(\frac{1}{6})$
Spinner #2	I $(\frac{1}{4})$	IT	AT	UT
	F $(\frac{3}{4})$	IF	AF	UF

The situation can also be represented with a tree diagram. In this model, the branching points indicate the events, and the branches stemming from each event indicate the possible outcomes for the event. For example, in the tree diagram at right the first branching point represents spinning the first spinner.

The first spinner can come out "I", "A" or "U", so each of those options has a branch. The number next to each letter is the probability that the letter occurs.

The numbers at the far right of the table represent the probabilities for each sequence of events. For example, $P(U \text{ and } T)$ can be found at the end of the bold branch of the tree. This probability, $\frac{1}{24}$, can be found by multiplying the fractions that appear on the bold branches.



Day 3

Explore: Students will work in groups on these problems.

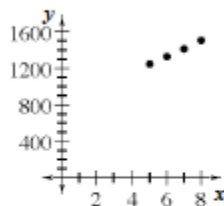
10-10. Eddie told Alfred, "I'll bet if I flip three coins I can get exactly two heads." Alfred replied, "I'll bet I can get exactly two heads if I flip four coins!" Eddie scoffed, "Well, so what? That's easier." Alfred argued, "No, it's not. It's harder." Who is correct? Show all of your work and be prepared to defend your conclusion.
[both equal $\frac{3}{8}$]

10-11. Use an area model or a tree diagram to represent the sample space for rolling two standard six-sided dice. [a: $\frac{2}{9}$, b: $\frac{1}{9}$, c: $\frac{2}{3}$]

- In the standard casino dice game the roller wins on the first roll if he rolls a sum of 7 or 11. What is the probability of winning on the first roll?
- The player loses on the first roll if he rolls a sum of 2, 3, or 12. What is the probability of losing on the first roll?
- If the player rolls any other sum, he continues to roll the dice until that sum comes up again or until he rolls a 7, whatever happens sooner. What is the probability that the game continues after the first roll?

10-12. What is the probability that $x^2 + 7x + k$ is factorable if $0 \leq k \leq 20$ and k is an integer? [$\frac{4}{21} \approx 19.05\%$; $k = 0, 6, 10, 12$ are factorable.]

- 10-13. Rondal High School had a student enrollment of 1245 in 2005, 1328 in 2006, and 1413 in 2007. School officials predicted that the 2008 enrollment would be 1505. The capacity of the school is 1800 students. [a: See graph below right. The equation of the line depends on the placement of the axes, but the slope is approximately 84; b: The slope represents the number of new students per year. Depending on the placement of axes, the y-intercept would logically be the number of students in year zero, when the school opened. c: It will get close in 2011, but if the annual increase is 85 to 90 students, it will reach capacity in 2012; d: Answers vary. 700-750, based on the 85 to 90 increase; e: The equation will not justify students' answers, because the school would not have been built with only a few students in it.]



- Graph this data. Determine the line of best fit and write a possible equation for that line.
 - What do the slope and y-intercept of the line represent?
 - Based on this information, when do you predict that the school will reach its capacity?
 - According to your graph and data, estimate the enrollment of Rondal High in the year 2000.
 - When do you think that Rondal High was built? Explain your thinking.
- 10-17. As you may remember, earthquake magnitudes are measured by the amount of energy that is released. Since the amount of energy released from a large earthquake can be millions of times greater than the energy released by a small quake, a scale was created (the **Richter scale**) to give magnitudes in numbers that are easy to use. An earthquake measuring 3.4 on the Richter scale, for example, releases $10^{3.4}$ kilojoules of energy. [a: 10, b: $10^{0.8} = 6.310$, c: 5.9]
- How many times more energy is released by an earthquake that measures 6.5 on the Richter scale than an earthquake that measures 5.5?
 - How many times more energy is released by an earthquake that measures 5.1 than an earthquake that measures 4.3? Give your answer both as a power of 10 and as a decimal.
 - What would the magnitude be of an earthquake that released half as much energy as an earthquake measuring 6.2 on the Richter scale?

10-18. A Nevada roulette wheel has 38 slots numbered 00, 0, 1, 2, 3, ..., 36. Eighteen of the numbers 1, 2, ..., 36 are red and 18 are black; 0 and 00 are green. [a: $\frac{9}{19} \approx 47.4\%$, b: $\frac{10}{19} \approx 52.6\%$, c: $(\frac{9}{19})^2 \approx 22.4\%$]



- What is the probability of landing on red?
- What is the probability of not landing on red?
- The bettor put his money on red twice in a row. Use an area or a tree diagram to show how to find the probability that he won both bets.

10-19. One way to win in a game with two dice is to roll a sum of six before getting a sum of seven. (Anything else that happens – sums of 2, 3, 4, 5, 8, 9, 10, 11, or 12 – are ignored.) [a: 5 ways, b: 6 ways, c: 11, d: $\frac{5}{11}$]

- How many ways are there to get a sum of six?
- How many ways are there to get seven?
- How many possible outcomes are important in this problem?
- What is the probability of getting a six before a seven?

10-20. Harold sorted his jellybeans into two jars. He likes purple ones best and the black ones next best, so these are both in one jar. His next favorites are yellow, orange, and white, and these are in another jar. He threw all the rest in the garbage. Harold allows himself to eat only one jellybean from each jar per day. He wears a blindfold when he selects his jellybeans so he cannot choose his favorites first. What is the probability that Harold gets one black jellybean and one orange jellybean, if the first jar has 60% black and 40% purple jellybeans and the second jar has 30% yellow, 50% orange, and 20% white jellybeans? [$0.6 \cdot 0.5 = 0.3 = 30\%$]

Share: The students will go over the problems as groups and explain their answers.

Summarize: I will make sure to go over the concepts the students missed.

Lesson 1 Post Test:

A jar contains 6 blue marbles, 7 red marbles, two green marbles, and ten yellow marbles.

- If you reach into the jar without looking, what is the probability of pulling out a red marble? a green marble?
- Davis pulled out a blue marble and kept it. Tess reached in. What is the probability she pulls out a blue marble as well? What is the probability that it is yellow instead?
- What is the probability of Aria reaching in and grabbing both green marbles?

[a: red, green; b: blue, yellow; c: or]

Unit 2: Probability and Counting

Lesson 2 Objective: The Fundamental Principle of Counting

Materials needed: paper/pencil

Homework: 10-63 to 10-68

Standard: compute basic statistics and understand the distinction between a statistic and a parameter.

Number of Days: 1

Launch: I will tell the students about the Minnesota license plates and we will see how many different options there are for them and if we need to make changes to the license plate to accommodate for our population.

Then we will look into other probability problems.

Phone numbers in the U.S. are composed of a three-digit area code followed by seven digits. License plates in some states are made up of three letters followed by a three-digit number. Postal ZIP codes are made up of five digits, and another four digits are often added. To win the lottery in one state you need to select the right five numbers from all the possible choices of five numbers out of 56. Consider these questions:



- How likely is it that you could win the lottery?
- Are there enough phone numbers for the exploding increase in cell phones?
- Jay wants to know the probability of randomly getting JAY on his license plates so he can avoid paying the extra amount for a special plate.

10-56. Nick came across the following problem: If a 4-digit number is randomly selected from all of the 4-digit numbers that use the digits 1, 2, 3, 4, 5, 6, and 7, with repeated digits allowed, what is the probability that the selected number is 2763? Nick knew that he had to figure out how many such numbers were possible, in order to know the size of his sample space. [a: It is too long. b: There are too many branches. c: Nick has to decide on each of four digits, 7 choices for each. d: There are 4 branch-points, 7 branches at each point. e: 7 for each, 2401; f: $\frac{1}{2401} \approx 0.0416\%$]

- Nick started to make a systematic list of the possibilities, but after the first few he gave up. What is the difficulty here in trying to create a list?
- Next he started a tree diagram. What problem did he encounter with the tree?
- He decided he needed a shortcut strategy for organizing this problem; otherwise he was going to be up all night. He started by asking himself, "How many decisions (about the digits) do I need to make?" With your team discuss his first question and then consider his next, "How many choices do I have for each decision (each digit)?"
- Audrey was working alone at home on the same problem at the same time. She was thinking of a tree diagram, when she asked herself, "How many branch-points will this tree have?" and "How many branches at each point?" Were these the same questions Nick was pondering?
- At the same moment they text messaged each other that they were stuck. When they talked, they realized they were on the same track. The problem asks for four-digit numbers, so there are four decisions. Simultaneously they said, "We need a decision chart."

$\overline{\text{1st digit}} \quad \overline{\text{2nd digit}} \quad \overline{\text{3rd digit}} \quad \overline{\text{4th digit}}$

How many choices are there for each decision? How many four digit numbers are there?

- What is the probably that the randomly selected number will be 2763?

Explore: Students will work in groups.

10-57. How many four-digit numbers could you make with the digits 1, 2, 3, 4, 5, 6, and 7 if you could not use any digit more than once in the four-digit number? Show a decision chart. Explain the similarities and differences between this situation and the one described in problem 10-56. [840, $7 \cdot 6 \cdot 5 \cdot 4$, answers will vary but should include: first number is the same, numbers decrease by one if numbers do not repeat, more choices if numbers repeat.]

10-58. Use a decision chart to answer each question below. [a: 504, b: 729, c: Both have three decision-points. The discs in part (a) mean each number can be used only once, but in part (b) you could spin the same number more than once. So in part (b) you have 9 choices every time, but for (a) the number of choices goes down after each choice.]

- A game contains nine discs, each with one of the numbers 1, 2, 3, 4, 5, 6, 7, 8, or 9 on it. How many different three-digit numbers can be formed by choosing any three discs, without replacing the discs?
- A new lotto game called "Quick Spin" has three wheels, each with the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 equally spaced around the rim. Each wheel is spun once, and the number the arrow points to is recorded. How many three-digit numbers are possible?
- Explain the similarities and differences between part (a) and part (b).

10-59. Marcos is selecting classes for next year. He plans to take English, physics, government, pre-calculus, Spanish, and journalism. His school has a six-period day, so he will have one of these classes each period. How many different schedules are possible? [720, a: $\frac{1}{6}$, b: $\frac{1}{30}$]



- What is the probability that Marcos will get first-period pre-calculus?
- What is the probability that Marcos will get both first-period pre-calculus *and* second-period physics?

10-60. On your calculator, find a button labeled $n!$ or $!$. On many graphing calculators, it is a function in the math menu and probability submenu. This is the **factorial** function button. [a: 40320, 5040, 720, 120, 24, 6, 2, 1; b: $6!$ c and d: These can be justified with a decision chart.]

- Find the value of 8 factorial (written $8!$), then $7!$, then $6!$, $5!$, ..., $1!$
- Which result is the same as the number of Marcos's possible schedules?
- What do you think $6!$ means? Why does it give the correct solution to the possible number of ways to arrange Marcos's schedule?
- Explain why $4!$ gives the correct solution to the possible number of ways to arrange the letters **M A T H**.

- 10-61. Remembering what $n!$ means can help you do some messy calculations quickly, as well as help you do problems that might be too large for your calculator's memory.

For instance, if you wanted to calculate $\frac{9!}{6!}$, you could use the $n!$ button on your calculator and find that $9! = 362,880$ and $6! = 720$, so $\frac{9!}{6!} = \frac{362880}{720} = 504$.

You could also use a simplification technique. Since $9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1!$ and $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, you can rewrite $\frac{9!}{6!} = \frac{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = 9 \cdot 8 \cdot 7 = 504$.

Use this simplification technique to simplify each of the following problems before computing the result. [a: 90, b: 190, c: 35, d: 405150]

a. $\frac{10!}{8!}$ b. $\frac{20!}{18!2!}$ c. $\frac{7!}{4!3!}$ d. $\frac{75!}{72!}$

Share: Each group will share one of the problems with the class.

Summarize: I'll make sure to cover additional topics the students did not get to.



METHODS AND MEANINGS

Conditional Probability, Independent Events

The probabilities you computed in problem 10-29 (Build-a-Farm) parts (d) and (e) are **conditional probabilities**. In part (d) you calculated the probability that you spun red given that you got a cow counter. $P(R|C)$ is shorthand notation for $P(\text{Red}|\text{Cow})$, or "probability of a red spin, given that you got a cow."

One way to compute this conditional probability is to write $P(\text{Red}|\text{Cow}) = P(\text{Red and Cow})/P(\text{Cow})$:

$$P(R|C) = \frac{P(R \text{ and } C)}{P(C)} = \frac{\frac{1}{6}}{\frac{13}{60}} = \frac{10}{60} \div \frac{13}{60} = \frac{10}{13}$$

In general, the probability of an event B, given that event A has already occurred is written:

$$P(B|A) = \frac{P(B \text{ and } A)}{P(A)}$$

Two events, A and B are **independent** when the probability of one does not depend on whether the other has occurred. In other words, the probability of B is the same as the probability of B, given A, or $P(B) = P(B|A)$.

Post Test:

- 10-62. A Scrabble® player has four tiles with the letters A, N, P, and S. [a: 24, b: The decision chart tells how many branches there are at each stage. c: $\frac{1}{4}$]
- How many arrangements of these letters are possible?
 - Draw a tree diagram that shows how to get the arrangements and explain how a decision chart represents the tree.
 - What is the probability of a two-year-old randomly making a word using the four letters?

Homework:

- 10-63. How many different batting orders can be made from the nine starting players on a baseball team? Write the answer using factorials and as a number. [$9! = 362,880$]
- 10-64. How many distinct rearrangements of the letters in the word FRACTIONS are there? [$9! = 362,880$]
- 10-65. Five students are running for Junior class president. They must give speeches before the election committee. They draw straws to see who will go first, second, etc. In how many different orders could they give their speeches? [$5! = 120$]
- 10-66. If $f(n) = n!$, evaluate each of the following ratios. [a: 20, b: 30, c: 36]
- $\frac{f(5)}{f(3)}$
 - $\frac{f(6)}{f(4)}$
 - $\frac{f(9)}{f(7)f(2)}$
- 10-67. Eight friends go to the movies. They want to sit together in a row with a student on each aisle. (Assume the row is 8 seats wide including 2 aisle seats.) [a: $8! = 40320$, b: $2 \cdot 7! = 10080$, c: $2 \cdot 6! = 1440$]



- How many ways can they sit in the row?
- If Kristen wants to sit in an aisle seat, how many ways can they all sit in the row?
- If Annabeth wants to sit on an aisle seat with Ian next to her, how many ways can the eight students sit?

- 10-68. In the casino game of Roulette, some players think that when the ball lands on red several times in a row that it will be more likely to land on black on the next spin. You can calculate to the conditional probability to demonstrate that this is not the case. In other words, the outcome of one spin of an honest roulette wheel does not have any effect on the outcome of the next one. [a: See diagram below right.

b: $\frac{18}{38} \cdot \frac{18}{38} = \frac{81}{361} \approx 22.44\%$, c: $\frac{81+81+9}{361} = \frac{171}{361} = \frac{9}{19} \approx 47.37\%$,

d: $\frac{81}{361} + \frac{171}{361} = \frac{81}{171} = \frac{9}{19} \approx 47.37\%$, e: They are the same; $\frac{9}{19}$]

- a. Make an area diagram for two spins of the wheel. Remember that there are 18 red numbers, 18 black numbers, and two greens.

- b. What is the probability that the ball will land on red twice in a row?

- c. Based on your diagram what is the total probability that the ball will land on red on the second spin? ($P(RR, BR, GR)$)

- d. Calculate the conditional probability that the first spin was red given that the second spin is red.

- e. How do your answers to parts (d) compare to the simple probability of the ball landing on red for any single spin? The fact that they are all the same means you have demonstrated that the probability of the ball landing on red is independent of where it landed before.

	R ($\frac{18}{38}$)	B ($\frac{18}{38}$)	G ($\frac{2}{38}$)
R ($\frac{18}{38}$)			
B ($\frac{18}{38}$)			
G ($\frac{2}{38}$)			

Unit 2: Probability and Counting

Lesson 3 Objective: Permutations

Materials needed: paper/pencil

Homework: 10-76 to 10-82 skip 10-81

Standard: compute basic statistics and understand the distinction between a statistic and a parameter.

Number of Days: 1

Launch: I will tell the kids about different arrangements there can be for a batting average. We will work through it together as a class. Then we will work on other permutation problems.

10-71. Jasper finally managed to hold on to some money long enough to open a savings account at the credit union. When he went in to open the account, the accounts manager told him that he needed to select a four-digit pin (personal identification number). She also said that he could not repeat a digit, but that he could use any of the digits 0, 1, 2, ..., 9 for any place in his four-digit pin. [a: 5040, b: Writing out the numbers in the factorials and reducing gives the decision-chart numbers $10 \cdot 9 \cdot 8 \cdot 7$.]

- How many pins are possible?
- Notice that the decision chart for this problem looks like the beginning of $10!$, but it does not go all the way down to 1. Factorials can be used to represent this problem, but you must compensate for the factors that you do not use, so you can write $\frac{10!}{6!}$. Discuss with your team how this method gives the same result as your decision chart.

Explore: Students will work in groups on these problems.

10-72. Twenty-five art students submitted sculptures to be judged at the county fair. Awards will be given for the six best sculptures. You have been asked to be the judge. You must choose and arrange in order the best six sculptures. [a: $25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 = 127,512,000$, b: $25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20$, c: $25 - 6 = 19$. Answers vary.]

- If you did this randomly, in how many ways could you choose the six best?
- Show how to simplify the expression: $\frac{25!}{19!}$.
- Where did the 19 come from in the original expression? How could you know the 19 from the numbers in the original problem? Why did $19!$ in the denominator become an important step in coming up with an answer to the problem? Discuss this with your team.

- 10-73. With your team, discuss how you could use factorials to represent each of the following situations. Then find the solutions. Four of the five problems involve permutations, and one does not. As you work, discuss with your team which problems fit the definition for permutations and why or why not.

[a: $\frac{52!}{(52-4)!} = 6,497,400$, b: $\frac{16!}{(16-7)!} = 57,657,600$,
c: $\frac{36!}{(36-3)!} = 42,840$, d: $35 \cdot {}_{35}P_2 = 35 \cdot \frac{35!}{33!} = 41,650$,
e: It is not a list of permutations because decisions can be repeated. $36^3 = 46,656$]



- Fifty-two contestants are vying for the Miss Teen pageant crown. In how many different ways can the judges pick the next Miss Teen and the runners-up one, two, and three?
- The volleyball team is sponsoring a mixed-doubles sand court volleyball tournament and sixteen pairs have signed up for the chance to win one of the seven trophies and cash prizes. In how many different ways can the teams be chosen and arranged for the top seven slots?
- Carmen is getting a new locker at school, and the first thing she must do is decide on a new locker combination. The three-number locker combination can be picked from the numbers 0 through 35. How many different locker combinations could she make up if none of the numbers can be repeated?
- How many three-digit locker combinations could Carmen make up if zero could only be the second or third number and none of the numbers can be repeated?
- How many locker combinations can Carmen have if she can use any of the numbers 0 through 35 and she can repeat numbers? Is this still a list of permutations?

Share: Students will share their answers.

Explore: Students will work in groups again.

- 10-74. Problems about batting orders and questions about how many numbers you could make without repeating any digits are called **permutations**. The following lists give some more examples for determining what is or is not a permutation. [a: The cards would allow no repetition, so would have many fewer possibilities: ABC, ACB, BAC, BCA, CAB, CBA; b: In the second case, order is not important. The first list is longer. c: No repetition, order is important. d: Make a decision chart and multiply the number of choices times one less, times one less, etc. up to the number of decisions.]

Permutations:

- i. All the arrangements a child can make on a line on the refrigerator door with three magnetic letters A, B, and C.
- ii. All the 4-digit numbers you could make using seven square tiles numbered 2, 3, 4, 5, 6, 7, and 8.
- iii. From a group of 8 candidates, one will become president, one vice president, and one secretary of the school senate.

Not-permutations:

- iv. All of the possible three letter license plates using A, B, and C.
 - v. The possible 4-digit numbers you could write if you could choose any digit from the numbers 2, 3, 4, 5, 6, 7, 8.
 - vi. From a group of 8 candidates, three will be selected to be on the spirit committee.
- a. Below is a list of all of the license plate letter triples that can be made with the letters A, B, and C.

AAA	BBB	CCC	AAB	ABA
BAA	AAC	ACA	CAA	ABB
BAB	BBA	ACC	CAC	CCA
ABC	ACB	CAB	BAC	CBA
BCA	BCC	CBC	CCB	CBB
BCB	BBC			

How is this list different from the list of arrangements of three cards with the letters A, B, and C printed on them? Explain and make the list of cards to demonstrate your ideas.

- b. Imagine two lists of three selected from 8 candidates (for descriptions *iii* and *vi* above). How do they differ? Which list would be longer?
- c. What are the important characteristics that a counting problem has to have in order to classify it as a permutations problem?
- d. Discuss with your team a *general* method for solving the examples above in the "Permutation" column and write a description for your general method that would work for any problem that could be identified as a permutations problem.

Share: Talk about the problems as a class.

Summarize: Make sure students get all main concepts.



METHODS AND MEANINGS

Fundamental Principle of Counting and $n!$

The **fundamental principle of counting** provides a shortcut for counting the branches of a symmetric tree diagram (finding the size the sample space) by multiplying.

For independent events, if an event A can occur in m different ways and an event B can occur in n different ways, then event A followed by event B can happen in $m \cdot n$ different ways. For a sequence of independent events, a **decision chart** is a useful tool.

For example: Ivan always has a sandwich for lunch. He may choose from three kinds of bread (white, wheat, or rye), four lunch meats (salami, ham, bologna, or pastrami), and two greens (lettuce or sprouts). Thus, he has three decisions to make: which bread, which lunchmeat, and which green. The decision chart for this situation is shown below.

$$\frac{3}{1st\ decision} \cdot \frac{4}{2nd\ decision} \cdot \frac{2}{3rd\ decision} = 24\ choices$$

Factorial is shorthand for the product of a list of consecutive, descending whole numbers from the largest down to 1:

$$n! = n(n-1)(n-2)\dots(3)(2)(1)$$

For example, 4 factorial or $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

Post Test:

- 10-75. For the homecoming football game the cheerleaders printed each letter of the name of your school's mascot on a large card. Each card has one letter on it, and each cheerleader is supposed to hold up one card. At the end of the first quarter, they realize that someone has mixed up the cards. [**Answers will depend on the name of the mascot.**]
- How many ways are there to arrange the cards?
 - If they had not noticed the mix up, what would be the probability that the cards would have correctly spelled out the mascot?

Homework:

10-76. Mr. Dobson is planning to give a quiz to his class tomorrow. Unfortunately for his students, Mr. Dobson is notorious for writing quizzes that seem to have no relevance to the subject. With this in mind, his students know that their efforts will be purely guesswork. If the quiz contains ten questions that the students will have to match with ten given answers, what is the probability that Rodney Random will get all ten questions matched correctly? [$\frac{1}{10!}$]

10-77. Which is bigger: $(5 - 2)!$ or $(5 - 3)!$? Justify your answer. [The first because $3! > 2!$]

10-78. Write an equivalent expression for each of the following situations that does not include the factorial (!) symbol. [a: $(n - 3)(n - 4)(n - 5)(n - 6)(n - 7)$; b: $n + 2, n + 1, n, n - 1, n - 2$; c: $n(n - 1)(n - 2)$; d: $(n + 2)(n + 1)n(n - 1)$]

- a. The first five factors of $(n - 3)!$ b. The first five factors of $(n + 2)!$
- c. $\frac{n!}{(n-3)!}$ d. $\frac{(n+2)!}{(n-2)!}$

10-79. What do you think $0!$ is equal to? [a: 1, b: $\frac{8!}{0!}$, If $0! = 0$, you would be dividing by 0. c: $\frac{3!}{3} = 2!$, $\frac{2!}{2} = 1!$, $\frac{1!}{1} = 0!$]

- a. Try it on your calculator to see what you get.
- b. What does ${}_8P_8$ mean? What *should* ${}_8P_8$ be equal to? Write ${}_8P_8$ using the factorial formula. Why is it necessary for $0!$ to equal 1?
- c. Do you remember how to show that $2^0 = 1$? You can use a sequence of powers of 2 ($\frac{2^4}{2} = 2^3$, $\frac{2^3}{2} = 2^2$, $\frac{2^2}{2} = 2^1$, so $\frac{2^1}{2} = 2^0$), but we also know that $\frac{2^1}{2} = 1$. Therefore $2^0 = 1$.

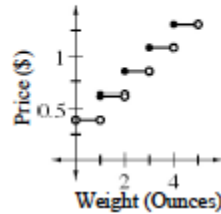
You can construct a similar pattern for $0!$, starting with $\frac{5!}{5} = 4!$ and then $\frac{4!}{4} = 3!$. Continue the pattern and make an argument to justify that $0! = 1$.

10-80. A state is chosen at random from the 50 states. Find the probability that the state meets each of the following criteria: [a: $\frac{14}{50}$, b: 1, c: 0, d: $\frac{27}{50}$]



- a. It is on the east coast.
- b. It has at least one representative in the House of Representatives.
- c. It has three U.S. senators.
- d. It does not border an ocean or the Gulf of Mexico.

- 10-82. In the year 2006, the Postal Service required 39 cents postage on a first-class letter weighing less than, but not equal to 1 ounce. An additional 23 cents was added for each ounce, or part of an ounce above that. Graph the relationship between price and weight for first-class mail in 2006. [See graph at right.]



Unit 2: Probability and Counting

Lesson 4 Objective: Combinations

Materials needed: paper/pencil

Homework: 10-89 to 10-98 skip 10-97

Standard: compute basic statistics and understand the distinction between a statistic and a parameter.

Number of Days: 1

Launch: I'll start out by talking about a card game the class likes. Then we will talk about their cards in their hands and how we are now talking about the groups of cards, not a specific arrangement of the cards. Then we will start a problem.

- 10-84. Five members of the Spirit Club have volunteered for the club governing board. These members are Al, Barbara, Carl, Dale, and Ernie. The club members will select three of the five as board members for the next year. One way to do this would be to elect a governing committee of three in which all members would have the same title. A second way would be to select a president, vice-president, and secretary.
- How many different lineups of officers are possible? This means a president, vice-president, and a secretary are chosen. Al as president, Barbara as vice-president, and Carl as secretary is considered a different possibility from Al as president, Barbara as secretary, and Carl as vice-president. [$5 \cdot 4 \cdot 3 = 60$]
 - How many different governing committees are possible? In this case, it is a good idea to make a list of all the possibilities, which are called combinations. [ABC, ABD, ABE, ACD, ACE, ADE, BCD, BCE, BDE, CDE; 10]
 - Felicia decides that she wants to volunteer as well. [*i*: $6 \cdot 5 \cdot 4 = 120$, *ii*: ABC, ABD, ABE, ABF, ACD, ACE, ACF, ADE, ADF, AEF, BCD, BCE, BCF, BDE, BDF, BEF, CDE, CDF, CEF, DEF; 20]
 - How many different possibilities for officers are possible now?
 - How many different governing committees are possible now? Again, make the list of all of the possibilities, or combinations.

- d. Since there are more volunteers, the spirit club has decided to appoint another committee member. [*i*: $6 \cdot 5 \cdot 4 \cdot 3 = 360$, *ii*: ABCD, ABCE, ABCF, ABDE, ABDF, ABEF, ACDE, ACDF, ACEF, ADEF, BCDE, BCDF, BCEF, BDEF, CDEF; 15]
- If they add a treasurer to the list of officers, how many different ways are there to select the four officers are possible?
 - If they choose a governing committee of four, how many possibilities are there?

Explore: Students will work in groups.

10-85. Compare the results you got for each set of numbers in problem 10-84 when the roles were determined (permutations) and when there were no specific roles (combinations). [**The number of combinations is 6 times greater for the 5 choose 3 situations and 24 times greater for the 6 choose 4.** **a: Conjectures vary. In situations of choosing r from a group of n , the number of permutations is $r!$ times the number of combinations.** **b: The number of combinations is 15 and the number of permutations is 30.** **c: ${}_n P_r = r!({}_n C_r)$ or ${}_n C_r = \frac{{}_n P_r}{r!}$]**

- Work with your team to develop a conjecture about the mathematical relationship between permutations and combinations chosen from the same sized groups. Be prepared to share your thinking with the class.
- Test your conjecture by calculating the number of permutations and combinations of 2 items chosen from 6. Does it work?
- How can you generalize your conjecture so that it can be applied to permutations and combinations of r items chosen from n ? Write a formula relating permutations (written ${}_n P_r$) and combinations (written ${}_n C_r$).

10-86. Now you will use your graphing calculator to test the formula you wrote in problem 10-85. Try 4 items chosen from 20. Does your formula work? [**a: ${}_{20} C_4 = \frac{{}_{20} P_4}{4!} = 4845$** , **b: Sample response: There are more permutations because of the different possible arrangements. To get combinations start with the number of permutations then divide out the number of arrangements.**]



- How many possibilities would you have to test to be sure that your formula is correct?
- With your team, find a way to justify the logic of your formula. How can you convince someone that it has to be correct for all numbers?

10-88. In the game of poker called “Five-Card Draw,” each player is dealt five cards from a standard deck of 52 cards. While players tend to arrange the cards in their hands, the order in which they get them does not matter. How many five-card poker hands are possible? Use the methods you developed in today’s investigation. [**2,598,960**]

Share: Students will share their answers.

Summarize: I’ll go over main points and review from last section.



METHODS AND MEANINGS

Permutations

Eight people are running a race. In how many different ways can they come in first, second, and third?

This is a problem of counting **permutations**, or arrangements, and the result can be represented ${}_8P_3$, which means the number of ways to **choose and arrange** three different (not repeated) things from a set of eight. There are several ways to write ${}_8P_3$:

$${}_8P_3 = 8 \cdot 7 \cdot 6 = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{8!}{5!} = \frac{8!}{(8-3)!} = 336$$

In general, ${}_nP_r = \frac{n!}{(n-r)!} = n(n-1)(n-2)\dots(n-r+1)$ for n items chosen r at a time.

Post Test:

- 10-87. In one state lottery there are 56 numbers to choose from. How many ways are there to choose 6 of the 56 numbers? How many combinations of 6 numbers can be chosen from the list of 56? Does the order in which the numbers are chosen matter? [**Order is not important.** a: 32,468,436; b: $\frac{1}{32,468,436}$]
- Find the number of possible combinations for the lottery.
 - What is the probability of selecting the six winning numbers?

Homework:

- 10-89. There are twelve people signed up to play darts during lunch. How many ways can a three-person dart team be chosen? [220]
- 10-90. Find the value of each permutation below. [a: 60, b: 840, c: 56, d: 720]
- ${}_5P_3 = ?$
 - ${}_7P_4 = ?$
 - ${}_8P_2 = ?$
 - ${}_{10}P_3 = ?$
- 10-91. The first four factors of $7!$ are 7, 6, 5, and 4 or 7, $(7-1)$, $(7-2)$, and $(7-3)$. [a: 12, $(12-1)$, $(12-2)$, $(12-3)$; b and c: $n(n-1)(n-2)(n-3)(n-4)(n-5)$]
- Show the first four factors of $12!$ in the same way the factors of $7!$ are shown above.
 - What are the first six factors of $n!$?
 - What is ${}_nP_6$?

- 10-92. On a six-person bowling team, only four players bowl in any game. How many different four-person teams can be made if the order in which they bowl does not matter? [15]

- 10-93. How many different bowling lineups of four players can be made in the previous problem if order does matter? [$6P_4 = 360$]



- 10-94. Here is another way to think about the question: "What is $0!$?"

- How many ways are there to choose all five items from a group of five items? What happens when you substitute into the factorial formula to compute ${}_5C_5$? Since you know (logically) what the result has to be, use this to explain what $0!$ must be equal to. [There is one way to choose all five. $\frac{5!}{5!0!} = 1$. In order to have the formula give a reasonable result for all situations, it is necessary to define $0!$ as equal to 1.]
- On the other hand, how many ways are there to choose *nothing* from a group of five items? And what happens when you try to use the factorial formula to compute ${}_5C_0$? [There is one way to choose nothing. $\frac{5!}{0!5!} = 1$]

- 10-95. In the card game called "Twenty-One," two cards are dealt from a randomly shuffled deck (a regular deck of 52 cards). The player's goal is to get the sum of his or her cards to be as close to 21 as possible, without going over 21. To establish the sample space for this problem, you need to think of choosing two from a set of 52. [a: ${}_{52}C_2 = 1326$, b: $\frac{16C_2}{52C_2} = \frac{120}{1326} \approx 0.09$, d: $\frac{12C_2}{52C_2} = \frac{66}{1326} \approx 0.0498$]



- How many ways are there to do this?
- The tens, jacks, queens, and kings all have a value of 10 points. There are four of each in a standard deck. How many ways are there to choose two cards with a value of 10 out of the 16 that are in the deck? What is the probability of being dealt two ten-valued cards?
- If you did not already do so, write your solution to part (b) in the form $\frac{aC_b}{dC_e}$.
- What is the probability of being dealt two face cards? (Face cards are Kings, Queens and Jacks, the cards that have faces.)
- If you did not already do so, write your solution to part (d) in the form $\frac{aC_b}{dC_e}$.

- 10-96. Given seven points in a plane, no three of which are collinear. [a: ${}_7C_2 = 21$; b: ${}_7C_3 = 35$; c: ${}_7C_4 = 35$; d: Choosing three points to form a triangle is the same as choosing four points to *not* be part of the triangle. Those four points form a quadrilateral, ${}_7C_3 = \frac{7!}{4!3!} = \frac{7!}{3!4!} = {}_7C_4$.]
- How many different lines are determined by these points?
 - How many distinct triangles can be formed?
 - How many distinct quadrilaterals can be formed?
 - Explain why the answers to parts (b) and (c) are the same.
- 10-98. Find the value of each of the following combinations. [a: 45, b: 792, c: 7]
- ${}_{10}C_8$
 - ${}_{12}C_7$
 - ${}_7C_1$

Sources

J. Kysh, L. Dietiker, T. Sallee, & B. Hoey. (2009). Algebra 2 Connections. College Preparatory Mathematics.

This lesson will take 5 days.

Objective: Students will identify a recursive and explicit formula for different patterns. The students will create a table and a graph from their patterns.

Minnesota State Standard:

8.2.2.4 Represent arithmetic sequences using equations, tables, graphs, and verbal descriptions, and use them to solve problems.

8.2.2.5 Represent Geometric sequences using equations, tables, graphs and verbal descriptions, and use them to solve problems.

Launch: My son is learning about patterns from his cousins. My parents have stacking cups that start out big and get smaller. They know the cups go in a pattern as well. They know they need to go Blue, Green, Yellow, red and then the pattern repeats. Can any of you think of places you see patterns or use patterns? Today we are going to look at patterns and see how they relate to something called an arithmetic sequence. Sequence can be another way we talk about a pattern.

Explore: Students will work with a partner and look at patterns and come up with what the next figure will look like. They will create a table, a graph, a recursive and explicit formula. See attached sheets labeled Explore 1 patterns.

Share: Students will share what they came up with for the recursive and explicit formulas for the different patterns. Students should also mention something about all the graphs that they graphed.

Something students should notice is all the explicit formulas are linear and all the graphs are linear.

Summarize: All these patterns made linear functions but since we could write all these patterns as an ordered list of numbers we could call the patterns sequences more specifically since the functions were linear we call this sequence an arithmetic sequence. Have students take down the attached note sheet found in the Big Ideas Algebra 1 book.

Students should make sure that they understand arithmetic sequences go from term to term by always adding. Students should hopefully start relating this to our linear functions and see that the common difference that we are adding or subtracting by is really our slope. Students should then start thinking about how they can find their y-intercept. Thinking of the definition of y-intercept can help you. If you noticed all our patterns started with 1 as our first x value. We could go find the previous value of $x = 0$ to find our y-intercept to be able to write an equation.

Examples to do before Exploration 2

What are the next 3 terms in the sequences

1.) . 1, 3, 5, 7, 9....

2.) 2,4,6,8.....

What is the explicit formula that goes with the following sequences. Also how could you find the 10th term. Think back to how you got your explicit formula you were able to multiply the common difference by the figure number and then add something so to find the 10th term you just put 10 in for x

3.) 3, 6, 9 ,12.....

4.) 1,5,9,13.....

Exploration 2: Students will work with a partner and look at the following arithmetic sequences to see if they can come up with the next terms and also see if they can come up with an explicit formula. See attached sheet labeled Exploration 2

Share: Students will share with the class what they thought the explicit formulas were and how they found them.

Summarize: I will remind students that arithmetic sequences go from one term to the next by always adding or subtracting.

Analyze/ Assess

For 8th graders they also need to know that an arithmetic sequence can start with $x = 0, 1, 2, \dots$ instead of $x = 1, 2, 3, \dots$ Like we have had. I would need to spend another day or 2 talking about that as well.

Exploration 3: We will continue to work with patterns this time though we aren't going to end up with linear equations we will end up with a different type of function. We are going to start by going through the paper folding activity (see attached sheet)

Share: Students will share what they found in the activity. Students will also share what their explicit and recursive formulas were.

Exploration 4: Students will work with a pattern on the yarn activity

Share: Students will again share what they found in the activity. Students will also share what their explicit and recursive formulas were.

Summarize: Hopefully students noticed this time they were getting exponential functions for their graphs and explicit equations. I would tell students this type of sequence is a geometric sequence. This sequence goes from one term to the next by multiplying by a common ratio.

Core Concept

Geometric Sequence

In a **geometric sequence**, the ratio between each pair of consecutive terms is the same. This ratio is called the **common ratio**. Each term is found by multiplying the previous term by the common ratio.



Students will need to see that a geometric sequence is in the form $f(x) = a(b)^x$ where b is the common ratio and a is the value of y when $x = 0$. Students should start to recognize that a geometric sequence goes from term to term by multiplying and an arithmetic sequence goes from term to term by adding or subtracting. As a class we will look at the following problems. What would the next 3 terms of the sequence be?

1. 3, 6, 12, 24, ...
2. 8, 4, 2, 1, $\frac{1}{2}$, ...

What would the explicit formula be for the following sequences could you find the 10th term. Remember we would have our equations in the form $y = ab^x$ where b is the common ratio and a is the y value when $x = 0$. If you remember from arithmetic sequences to find the 10th term you can just plug 10 in for x .

1. -2, 4, -8, 16, ...
2. 27, 9, 3, 1, $\frac{1}{3}$, ...

Exploration 5:

With a partner students will go through the following problems found on the attached sheet labeled Exploration 5

Share: students will share with the class what they found as their answers and how they came up with their formulas.

Summarize: Remind students that Geometric Sequences go from one term to the next by always multiplying by a common ratio.

Analyze/Assess:

A **sequence** is an ordered list of numbers. Each number in a sequence is called a **term**. Each term a_n has a specific position n in the sequence.



Core Concept

Arithmetic Sequence

In an **arithmetic sequence**, the difference between each pair of consecutive terms is the same. This difference is called the **common difference**. Each term is found by adding the common difference to the previous term.



EXAMPLE 1 Extending an Arithmetic Sequence

Write the next three terms of the arithmetic sequence.

$$-7, -14, -21, -28, \dots$$

SOLUTION

Use a table to organize the terms and find the pattern.

Position	1	2	3	4
Term	-7	-14	-21	-28

$$+(-7) \quad +(-7) \quad +(-7)$$

Each term is 7 less than the previous term. So, the common difference is -7 .

Add -7 to a term to find the next term.

Position	1	2	3	4	5	6	7
Term	-7	-14	-21	-28	-35	-42	-49

$$+(-7) \quad +(-7) \quad +(-7)$$

Explore 1 patterns.

Find the next three figures. When you are finished create a table, a graph, a recursive formula, and an explicit formula.

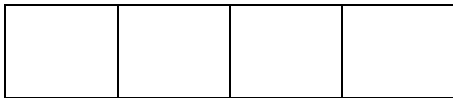
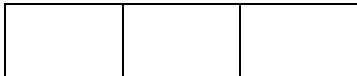
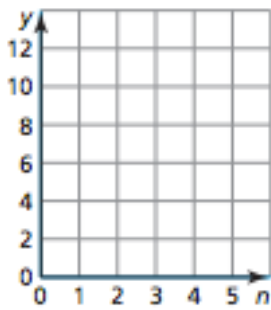


Figure	Number Of tiles
1	
2	
3	
4	
5	

Recursive formula:

Explicit Formula

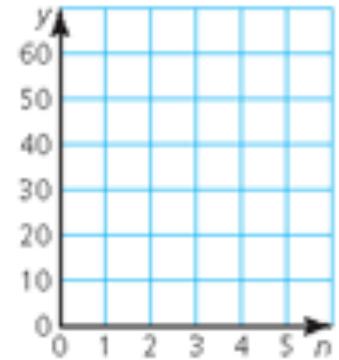


For a, b, and c complete the tables and graph the table then find recursive and explicit formulas

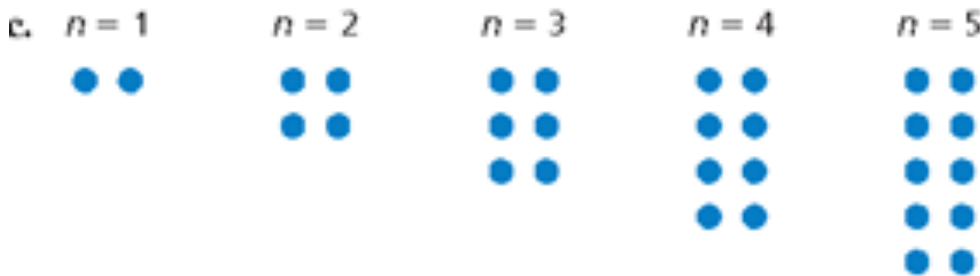
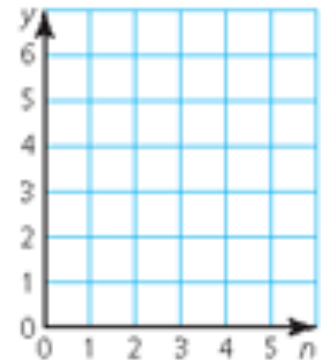
Work with a partner. Use the figures to complete the table. Plot the points given by your completed table. Describe the pattern of the y-values.



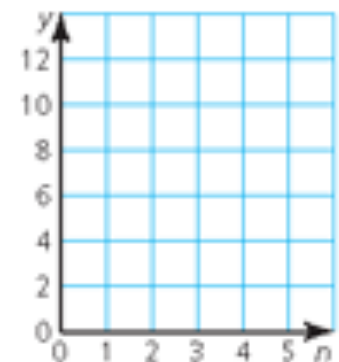
Number of stars, n	1	2	3	4	5
Number of sides, y					



n	1	2	3	4	5
Number of circles, y					



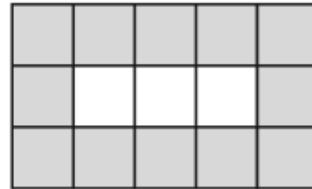
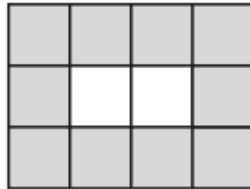
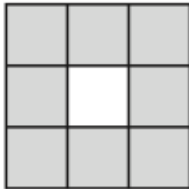
Number of rows, n	1	2	3	4	5
Number of dots, y					



To answer this problem it will help to make the next figure and make a table of values and then come up with a recursive and explicit formula.

Answer the following questions about sequences.

1. The figures below show that 1 square requires 8 squares to surround it, 2 squares require 10, and 3 squares require 12. How many squares will it take to surround 10 squares? 100 squares?



# of squares inside	# of squares surrounding
1	8
2	10
3	12

Recursive Formula:

Explicit Formula:

Exploration 2

In Exercises 1–6, write the next three terms of the arithmetic sequence.

1. 1, 8, 15, 22, ...

2. 20, 14, 8, 2, ...

3. 12, 21, 30, 39, ...

4. 5, 12, 19, 26, ...

5. 3, 7, 11, 15, ...

6. 2, 14, 26, 38, ...

**In Exercises 16–21, write an equation for the n th term of the arithmetic sequence.
Then find a_{10} .**

16. $-5.4, -6.6, -7.8, -9.0, \dots$

17. $43, 38, 33, 28, \dots$

18. $6, 10, 14, 18, \dots$

19. $-11, -9, -7, -5, \dots$

Come up with 4 of your own arithmetic sequences. Can you find the next 3 terms and a rule for the arithmetic sequence.

Paper-Folding Activity

Part I

- 1) Look at your sheet of paper and determine the number of “sections” the paper has when it is completely unfolded. Record this data in the table below.
- 2) Fold your piece of paper in half and determine the number of sections the paper has after you have made a fold. Record this data in the table below.
- 3) Fold the piece of paper in half again and determine the new number of sections. Record your data in the table. Continue in the same manner, folding the paper and recording the data, until it becomes too hard to fold the paper any more. (This will probably happen around the 6th fold.)

Paper-Folding Data

Number of Folds	Number of Sections
0	

- 4) Which column of your table is the input? Which column is the output?
- 5) Using graph paper, make a scatter plot of the data.
- 6) By studying the graph and/or table, predict what you think the number of sections will be for 10 folds.
- 7) Determine a mathematical model (equation) that represents this data by examining the patterns in the table. I want both a recursive and explicit formula.



Yarn-Cutting Activity

Part II

- 1) Begin with one piece of yarn and consider it to be one unit long. This data is recorded in the table below.
- 2) Fold your piece of yarn in half and cut it. You should now have two pieces of yarn, each half the length of the original piece. This information has also been recorded for you in the table below.
- 3) Continue in the same manner, cutting each piece of yarn in half and recording the new number of pieces and new length, until it becomes too hard to cut the yarn. (This will probably happen around the 5th cut.) Be sure to record your data in the table.

Yarn-Cutting Data

Number of Pieces	Length of Each New Piece
1	1
2	$\frac{1}{2}$

- 3) Which column of your table is the input? Which column is the output?
- 4) Using graph paper, make a scatter plot of the data.
- 5) By studying the graph and/or table, predict what you think the length of each new piece will be after the 10th cut.
- 6) Determine a mathematical model (equation) that represents this data by examining the patterns in the table. I want both an explicit and recursive formula.

Exploration 5

In Exercises 7–9, write the next three terms of the geometric sequence.

- 7.) 7, 21, 63, 189, ... 8.) 576, 288, 144, 72, ... 9.) 5, -10, 20, -40, ...

In Exercises 13–20, write an equation for the n th term of the geometric sequence. Then find a_6 .

13. 6561, 2187, 729, 243, ... 14. 8, -24, 72, -216, ... 15. 3, 15, 75, 375, ...

16.

n	1	2	3	4
a_n	2916	972	324	108

17.

n	1	2	3	4
a_n	11	44	176	704

Come up with 3 Geometric sequences for your partner to try. Have your partner find the next 3 terms and a formula for the sequence.

Pre/Post Test for Arithmetic and geometric sequences

Are the Sequences below Arithmetic or Geometric? Can you write a recursive and explicit formula for the sequence? Can you find the 10th term? We will say $x = 1, 2, 3, \dots$

1. 3, 6, 9, 12.....

2. 32, 16, 8, 4, 2, 1.....

3. 12, 8, 4, 0, -4.....

4. -3, 6, -12, 24, -48.....

Unit 4

Objective: students will see if they can come up with all the squares in an $n \times n$ grid.
This lesson will take 1 day.

Standard: NCTM problem solving standard.

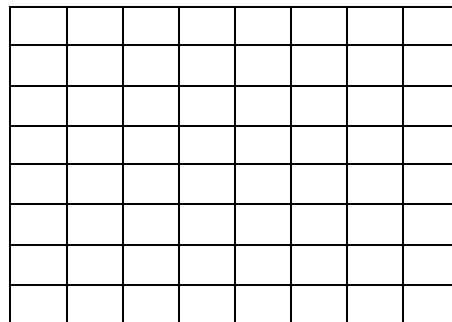
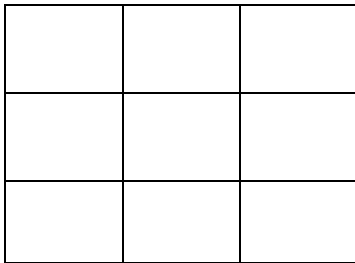
This lesson is from Interactive mathematics program IMP year 1 book

Launch: To get students to start problem solving I would have students solve the triangle problem. Each edge needs to add up to 12 and you can only use the numbers 1 – 9. The numbers would go where the x's are.

```

  X
 X  x
X   x  x
```

Explore: Looking at the 8×8 square checkerboard below, can you find how many squares are on it? It may be easier to start by looking at a 3×3 square grid. It may be best to next look at a 4×4 grid and then a 5×5 grid before you jump to an 8×8 square checkerboard.



Once students start working you may want to add to the question. How many squares of various sizes are on an 8 by 8 checkerboard altogether?

When you are confident that you have counted all the squares on an 8 by 8 checkerboard, move on to the generalization in Question 2.

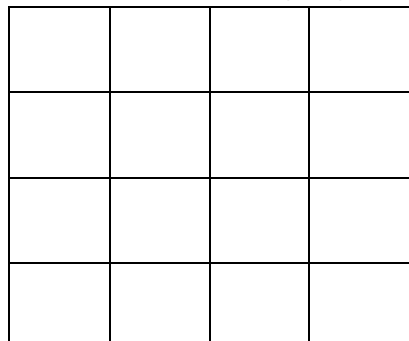
2. Suppose you have a square checkerboard of some other size (not 8 by 8) How can you determine how many square are on it altogether?

You will know you are done when no matter what size checkerboard you are given, you can give a clear procedure for easily computing the total number of squares.

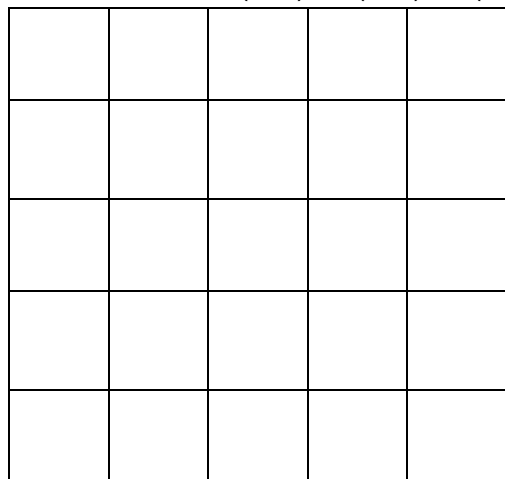
Share: Students will share how many squares they found on the 3 x 3 grid, the 4x4 grid, the 5 x 5 grid, and the 8x8 grid.

3x3 answer: There is 1 big square + 4 2x2 squares + 9 little squares. = 14 total squares

4x4 answer : $1 + 4 (3 \times 3) + 9 (2 \times 2) + 16 (1 \times 1) = 30$



5 x 5 answer: $1 (5 \times 5) + 4(4 \times 4) + 9(3 \times 3) + 16(2 \times 2) + 25(5 \times 5)$



Students may start seeing a pattern they may notice that they are adding up all the perfect squares until they hit the area of the square.

Students may see the answer to an 8 x 8 grid would be $1 + 4 + 9 + 16 + 25 + 36 + 49 + 64$ before even doing the problem. They may need to try all the grids up to 8.

The general rule for an $n \times n$ grid would be add up all the perfect squares $1 + 4 + 9 + 16 \dots + (n \times n)$

Summarize: This lesson was a problem solving lesson but just think of all the different concepts that we talked about: area, perfect squares, the definition of a square, adding, and trying to see a pattern.

Analyze/Assess

Objective: Be able to find the area of irregular shapes.
(1 day)

Standards: NCTM problem solving standard

Launch: If I wanted to carpet a room that was not a regular shaped room how could I find the area to know how much carpet I need?

Explore: Every student needs a Geo-board or Geo-board paper. Students need to make some kind of shape and see if they can find a relationship between the area, total pegs, border pegs, and inside pegs. Students will then make an intense shape and try to find the area, total pegs used, border pegs, and inside pegs. Students then need to write their data on the board so their classmates can see it. Once your data is on the board look at your classmates data as well and see if you can come up with some type of formula to get the area of your shape faster than just counting the area.

Share: Every student will come and put their data on the board. Students will share their thinking in coming up with a formula to find the area of irregular shapes.

Summarize: Hopefully after quite awhile of class discussion and students going back and forth we can come up with the formula $A = B/2 + I - 1$ I would then tell students this is actually called Pick's Formula.

Analyze/Assess

Unit 5-Winner, Winner?

Grade Level: Could be done with elementary, but I am using this for 8th grade.

Materials: Worksheet at the bottom of page, pre-test, and post-test, computers for research

Objective: Students will learn four different methods to vote with ranking.

Standard: There are no Minnesota State Standards for this unit.

Day 1:

Launch: As an 8th grade advisor I must have you vote for a President, a Vice President, and a Homecoming Paige. How do you think we should go about voting for each position? Have a discussion about what students come up with and prepare to vote for each candidacy.

Explore: Have students take a look at the following table.

Candy Bar	Ranking	Ranking	Ranking	Ranking	Ranking	Ranking
# of people						

Have the students group up into groups of 3-4 students and have them as a group come up with their favorite type of candy bar. Fill in the table with three different types. Then have the students grab a sheet of paper and vote for their favorites by ranking of 1-favorite, 2-second favorite, and 3-third favorite. Give the students some time to complete this. When they have a ranking, come up to the board and write it down. There should be only 6 different ways to rank them, so if there are not all six, have the students figure out the rest. When this is done, could the number of students with each ranking below the table. Have a discussion about which candy bar won? Let them know that tomorrow they will be researching different voting methods and teaching the rest of the class how to use the method and using the information above, decide the winning candy bar.

Day 2, 3, & 4:

Give each group a voting method to research and have them present the method to the class and show which candy bar won. The methods are 1) Plurality Win, 2) Borda Count Method, 3) Head to Head, and 4) Plurality with Elimination. Students will then as a group go and do research about their method and figure out a way to teach the method and explain which candy bar won using their method. If you have more than 4 groups, double up on a couple of them.

Share: Students will teach their method to the class and then discuss which one won using their method. After all the groups have finished, have a discussion about which method was the best and for what circumstances.

Day 5, 6, & 7: Review each of the different voting methods that were discussed the days before. Hand out the worksheet and have students go through in different groups. Try and have one person from each of the different methods in a new group, so they can really help each other if they have questions. After each group is done, have four different groups come up to the board and show their results of one method.

Have the groups come up with a survey to give the class about whatever topic they would like. Before they get the results, they need to figure out which method to use to find the winner or winners. They will then present to the class about their findings.

Day 8: Now that all the methods have been seen and hopefully at this point learned, let's talk about the original problem. We need to vote for president, vice-president, and homecoming paige. The big question before we vote is, which method should we use to see who won each position? Decide as a group. Then vote. Have the students figure out who won, but also have them figure out the other three methods they did not choose to see what happened with those.

Summarize: As a teacher, you are there to help the students after the launch. Help students find good resources for them to use and make sure students understand what their method is talking about and how to use it, so they can teach the others in the class. Try and limit this to three items. If you use more, allow more time.

Voting Methods worksheet

Rank the following soda pop in order from the most favorite (1) to the least favorite (3)

Mountain Dew _____

Coke _____

Pepsi _____

Results Table

Dew						
Coke						
Pepsi						

of people

Plurality-

Borda Count Method-

Plurality with Elimination-

Dew						
Coke						
Pepsi						

Head-to-Head-

	Dew	Coke	Pepsi
Dew			
Coke			
Pepsi			

Voting Methods Pre-Test

Rank the following past times in order from the most favorite (1) to the least favorite (3)

Video Gaming _____
 Watching TV _____
 Hanging with friends _____

Results Table

Gaming						
TV						
Friends						

of people

Plurality-

Borda Count Method-

Plurality with Elimination-

Gaming						
TV						
Friends						

Head-to-Head-

	Gaming	TV	Friends
Gaming			
TV			
Friends			

Voting Methods Post-Test

Rank the following past times in order from the most favorite (1) to the least favorite (3)

Video Gaming _____
 Watching TV _____
 Hanging with friends _____

Results Table

Gaming						
TV						
Friends						

of people

Plurality-

Borda Count Method-

Plurality with Elimination-

Gaming						
TV						
Friends						

Head-to-Head-

	Gaming	TV	Friends
Gaming			
TV			
Friends			